

Exact Solutions for Non-Hermitian Dirac-Pauli Equation in an intensive magnetic field

V.N.Rodionov

Plekhanov Russian University, Moscow, Russia, *E-mail*
vnrodionov@mtu-net.ru

Abstract

The modified Dirac-Pauli equations, which are introduced by means of γ_5 -mass factorization of the ordinary Klein-Gordon operator, are considered. We also take into account the interaction of fermions with the intensive homogenous magnetic field focusing attention to their (g-2) gyromagnetic factor. The basis of this approach is developing of methods for study of the structure of regions of unbroken \mathcal{PT} symmetry of Non-Hermitian Hamiltonians which be no studied earlier. For that, without the use of perturbation theory in the external field the exact energy spectra are deduced with regard to spin effects of fermions. We also investigate the unique possible of experimental observability the non-Hermitian restrictions in the spectrum of mass consistent with the conjecture Markov about Maximal Mass. This, in principal will may allow to find out the existence of an upper limit value in spectrum masses of elementary particles and confirm or deny the significance of the Planck mass.

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1 Introduction

Now it is well-known fact, that the reality of the spectrum in models with a non-Hermitian Hamiltonian is a consequence of \mathcal{PT} -invariance of the theory, i.e. a combination of spatial and temporary parity of the total Hamiltonian: $[H, \mathcal{PT}]\psi = 0$. When the \mathcal{PT} symmetry is unbroken, the spectrum of the quantum theory is real. This surprising results explain the growing interest

in this problem which was initiated by Bender and Boettcher's observation Ref.[1]. For the past a few years has been studied a lot of new non-Hermitian \mathcal{PT} -invariant systems Ref.[2] - [26].

The non-Hermitian \mathcal{PT} -symmetric γ_5 -extension of the Dirac equation is first studied in Ref.[2] and further illustrated in Ref.[3] and Ref.[4]-[9]. The purpose of this paper is the continuation of the studying examples of pseudo-Hermitian relativistic Hamiltonians, investigations of which was started by us earlier Ref.[5]-[9].

Here we are producing our investigation of non-Hermitian systems with γ_5 -mass term extension taking into account the external magnetic field. We also are studying the spectral and polarization properties of such systems and with this aim we start to consider solutions of the modified Dirac equation for free fermions. After that we take into account interaction with the intensive magnetic fields of charged and neutral particles having anomalous magnetic moments (AMM). The novelty of developed by us approach is associated with predictions of new phenomena caused by a number of additional terms of the non-Hermitian Hamiltonians, which radically changes the picture of interactions. But it is not only refers to the processes with large energies, but may be observed in the region of low energies when one takes into account the interaction AMM of fermions with intensive magnetic fields.

This paper has the following structure. In section II the non-Hermitian approach to the construction of plane wave solutions is formulated for the case free massive particles. In the third section we study the basic characteristics of modified Dirac models with γ_5 -massive contributions in the external magnetic field. Then, in the fourth section, we consider the modified Dirac-Pauli model in the magnetic field with non-Hermitian extension. This section also contains the discussion of the effects of the possible observability of the parameters m_1, m_2 taking into account the interaction of charged fermions together with regard to their AMM with the magnetic field. The fifth section contains Summary and Conclusions.

2 Non-Hermitian extensions of plane waves

Let us now consider the solutions of modified Dirac equations for free massive particles using the γ_5 -factorization of the ordinary Klein-Gordon operator. In this case similar to the Dirac procedure one can represent the Klein-Gordon operator in the form of a product of two commuting matrix operators:

$$\left(\partial_\mu^2 + m^2\right) = \left(i\partial_\mu\gamma^\mu - m_1 - \gamma_5 m_2\right)\left(-i\partial_\mu\gamma^\mu - m_1 + \gamma_5 m_2\right), \quad (1)$$

where designations $\hbar = c = 1$ are used and the physical mass of particles m is expressed through the parameters m_1 and m_2

$$m^2 = m_1^2 - m_2^2. \quad (2)$$

For so the function would obeyed to the equations of Klein-Gordon

$$\left(\partial_\mu^2 + m^2\right)\tilde{\psi}(x, t) = 0 \quad (3)$$

one can demand that it also satisfies one of equations of the first order

$$\left(i\partial_\mu\gamma^\mu - m_1 - \gamma_5 m_2\right)\tilde{\psi}(x, t) = 0 \quad \left(-i\partial_\mu\gamma^\mu - m_1 + \gamma_5 m_2\right)\tilde{\psi}(x, t) = 0 \quad (4)$$

Equations (4) of course, are less common than (3), and although every solution of one of the equations (4) satisfies (3), reverse approval has not designated. It is also obvious that the Hamiltonians, associated with the equations (4), no are Hermitian, because in it the γ_5 -dependent mass components appear ($H \neq H^+$):

$$H = \vec{\alpha}\mathbf{p} + \beta(m_1 + \gamma_5 m_2) \quad (5)$$

and

$$H^+ = \vec{\alpha}\mathbf{p} + \beta(m_1 - \gamma_5 m_2). \quad (6)$$

Here matrices $\alpha_i = \gamma_0 \cdot \gamma_i$, $\beta = \gamma_0$, $\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3$.

It is easy to see from (2) that the mass m , appearing in the equation (3) is real, when the inequality

$$m_1^2 \geq m_2^2. \quad (7)$$

is accomplished.

A. Mustafazadeh identified the necessary and sufficient requirements of reality of eigenvalues for pseudo-Hermitian and \mathcal{PT} -symmetric Hamiltonians and formalized the use these Hamilton operates in his papers Ref.[10], Ref.[11] and Ref.[19]-[25]. According to the recommendations of this works we can define Hermitian operator η , which transform non-Hermitian Hamiltonian

by means of invertible transformation to the Hermitian-conjugated one. It is easy to see that with Hermitian operator

$$\eta = e^{\gamma_5 \vartheta}, \quad (8)$$

where $\vartheta = \text{arctanh}(m_2/m_1)$ we can obtain

$$\eta H \eta^{-1} = H^+, \quad (9)$$

In addition, multiplying the Hamilton operator H from left to $e^{\vartheta \gamma_5/2}$ and taking into account that matrices γ_5 commute with matrices α_i and anti-commute with β , we can obtain

$$e^{\gamma_5 \vartheta/2} H = H_0 e^{\gamma_5 \vartheta/2}, \quad (10)$$

where $H_0 = \alpha p + \beta m$ is a ordinary Hermitian Hamiltonian of a free particle.

The mathematical sense of the action of the operator (8) it turns out, if we notice that according to the properties of γ_5 matrices, all the even degree of γ_5 are equal to 1, and all odd degree are equal to γ_5 . Given that $\cosh(x)$ decomposes on even and $\sinh(x)$ odd degrees of x , the expressions (9)-(10) can be obtained by representing non-unitary exponential operator η in the form

$$\eta = e^{\gamma_5 \vartheta} = \cosh \vartheta + \gamma_5 \sinh \vartheta, \quad (11)$$

where

$$\cosh \vartheta = m_1/m; \quad \sinh \vartheta = m_2/m. \quad (12)$$

The region of the unbroken \mathcal{PT} -symmetry of (5) can be found in the form (7). However, it is not apparent that the area with undisturbed \mathcal{PT} -symmetry defined by such a way does not include the regions, corresponding to the some unusual particles, description of which radically distinguish from traditional one.

If we used the standard representation of γ -matrixes the non-Hermitian Hamiltonian H can be whiten in the following matrix form

$$H = \begin{pmatrix} m_1 & 0 & p_3 - m_2 & p_1 - ip_2 \\ 0 & m_1 & p_1 + ip_2 & -m_2 - p_3 \\ m_2 + p_3 & p_1 - ip_2 & -m_1 & 0 \\ p_1 + ip_2 & m_2 - p_3 & 0 & -m_1 \end{pmatrix},$$

where p_i are components of momentum.

It is clear that

$$H\tilde{\psi} = E\tilde{\psi}.$$

The condition $\det(H - E) = (-E^2 + m_1^2 - m_2^2 + p_\perp^2 + p_3^2)^2 = 0$ results in the eigenvalues of E which are represented in the form:

$$E = \pm \sqrt{m_1^2 - m_2^2 + p_\perp^2 + p_3^2}, \quad (13)$$

where $p_\perp = \sqrt{p_1^2 + p_2^2}$ and $m_1^2 - m_2^2 = m^2$, that coincide with the eigenvalues of energy of Hermitian operator H_0 .

Let us now consider the state of a free particle with certain values of the momentum and energy, which is described by a plane wave and can be written as

$$\tilde{\psi} = \frac{1}{\sqrt{2E}} \tilde{u} e^{-ipx}. \quad (14)$$

It is easy to see that the wave amplitude \tilde{u} is determined by bispinor, normalization of which now needs an additional explanation.

Really using (4) and taking into account properties of matrices $\gamma_0, \vec{\gamma}, \gamma_5$, we can write also complex-conjugate equation

$$\left(-p_0 \tilde{\gamma}_0 - \mathbf{p} \tilde{\vec{\gamma}} - m_1 - \gamma_5 m_2\right) \tilde{\psi}^* = 0, \quad (15)$$

where $\tilde{\gamma}_\mu$ are transpose matrix. Rearranging function $\tilde{\psi}^*$ and introducing new bispinor $\overline{\tilde{\psi}} = \tilde{\psi}^* \gamma_0$, we can obtain

$$\overline{\tilde{\psi}} (\gamma p + m_1 - \gamma_5 m_2) = 0. \quad (16)$$

The operator p is assumed here acts on the function, standing on the left of it. Using (8) we can write equation (4), (16) in the following form

$$(p\gamma - m\eta) \tilde{\psi} = 0 \quad (17)$$

$$\overline{\tilde{\psi}} (p\gamma + m\eta^{-1}) = 0 \quad (18)$$

Multiplying (17) on the left of the $\tilde{\psi} e^{-\vartheta\gamma_5}$ and the equation (18) on the right of the $e^{\vartheta\gamma_5} \tilde{\psi}$ and summing up the resulting expressions, one can obtain

$$\tilde{\psi} e^{-\vartheta\gamma_5/2} \gamma_\mu e^{\vartheta\gamma_5/2} (p_\mu \tilde{\psi}) + (p_\mu \overline{\tilde{\psi}}) e^{-\vartheta\gamma_5/2} \gamma_\mu e^{\vartheta\gamma_5/2} \tilde{\psi} = p_\mu \left(\tilde{\psi} e^{-\vartheta\gamma_5/2} \gamma_\mu e^{\vartheta\gamma_5/2} \tilde{\psi} \right) = 0 \quad (19)$$

Here brackets indicate which of the function are subjected to the action of the operator p_μ . The obtained equation has the form of the continuity equation

$$\partial_\mu j_\mu = 0, \quad (20)$$

where

$$j_\mu = \widetilde{\psi} e^{-\vartheta \gamma_5/2} \gamma_\mu e^{\vartheta \gamma_5} \widetilde{\psi} = \left(\widetilde{\psi}^* e^{\vartheta \gamma_5} \widetilde{\psi}, \widetilde{\psi}^* \gamma_0 \vec{\gamma} e^{\vartheta \gamma_5} \widetilde{\psi} \right) \quad (21)$$

Thus here the value of j_μ is a 4-vector of current density of particles in the model with γ_5 -mass extension. It is very important that its temporal component

$$j_0 = \widetilde{\psi}^* e^{\vartheta \gamma_5} \widetilde{\psi} \quad (22)$$

does not change in time (see (20) and positively defined. It is easy to see from the following procedure. Let us construct the norm of any state for considered model for arbitrary vector, taking into account the weight operator η (11):

$$\widetilde{\psi} = \begin{pmatrix} x + iy \\ u + iv \\ z + iw \\ t + ip \end{pmatrix}.$$

Using (12), (22), in a result we have

$$\widetilde{\psi}^* \eta = \left(\frac{m_1 + m_2}{m} (x - iy), \frac{m_1 + m_2}{m} (u - iv), \frac{m_1 - m_2}{m} (z - iw), \frac{m_1 - m_2}{m} (t - ip) \right).$$

Then

$$\langle \widetilde{\psi}^* \eta | \widetilde{\psi} \rangle = \frac{m_1 + m_2}{m} (x^2 + y^2) + \frac{m_1 + m_2}{m} (u^2 + v^2) + \frac{m_1 - m_2}{m} (z^2 + w^2) + \frac{m_1 - m_2}{m} (t^2 + p^2) \quad (23)$$

is explicitly non negative, because $m_1 \geq m_2$ in the area of unbroken \mathcal{PT} -symmetry (7).

With the help of (17),(18) and properties commutation of γ -matrix one can obtain that components of new bispinor amplitudes must satisfy the following system of algebraic equations:

$$(\gamma p - m e^{\gamma_5 \vartheta}) \widetilde{u} = 0; \quad (24)$$

$$\widetilde{\bar{u}} (\gamma p - m e^{-\vartheta \gamma_5}) = 0, \quad (25)$$

where $\bar{\tilde{u}} = \tilde{u}^* \gamma_0$.

According to (8),(10) one can write bispinor amplitudes in the form

$$\tilde{u} = \sqrt{2m} \begin{pmatrix} A_1 w \\ A_2 w \end{pmatrix}; \quad (26)$$

$$\bar{\tilde{u}} = \sqrt{2m} \begin{pmatrix} A_1 w^*, & -A_2 w^* \end{pmatrix}, \quad (27)$$

where the notations are used:

$$A_1 = \cosh \frac{\vartheta}{2} \cosh \frac{\beta}{2} + \sinh \frac{\vartheta}{2} \sinh \frac{\beta}{2} (\mathbf{n} \vec{\sigma});$$

$$A_2 = \sinh \frac{\vartheta}{2} \cosh \frac{\beta}{2} + \cosh \frac{\vartheta}{2} \sinh \frac{\beta}{2} (\mathbf{n} \vec{\sigma}).$$

In addition, we have relations (11),(12) and the parameters $\cosh \beta = E/m$, $\sinh \beta = p/m$. And also w - two-component spinor, satisfying the normalization condition

$$w^* w = 1.$$

Besides need to note that $\vec{\sigma}$ are ordinary 2×2 -Pauli matrices and $\mathbf{n} = \mathbf{p}/p$ - a unit vector in the direction of the momentum.

The explicit form of these spinors can be found using the condition that spiral states correspond to the plane wave in which spinors w is a eigenfunctions of the operator $(\vec{\sigma} \mathbf{n})$

$$\vec{\sigma} \mathbf{n} w^\zeta = \zeta w^\zeta.$$

Therefore we get

$$w^1 = \begin{pmatrix} e^{-i\varphi/2} \cos \theta/2 \\ e^{i\varphi/2} \sin \theta/2 \end{pmatrix}, \quad w^{-1} = \begin{pmatrix} -e^{-i\varphi/2} \sin \theta/2 \\ e^{i\varphi/2} \cos \theta/2 \end{pmatrix},$$

where θ and φ - polar and azimuthal angles that determine the direction \mathbf{n} concerning to the axes x_1, x_2, x_3 .

It is easy to verify by the direct multiplication that

$$\bar{\tilde{u}} \tilde{u} = 2m.$$

This result however in advance obvious, because there are the connection between bispinor amplitudes of modified equations $\bar{\tilde{u}}, \tilde{u}$ and corresponding solutions of the ordinary Dirac equations:

$$\tilde{u} = e^{-\gamma_5 \vartheta/2} u$$

$$\bar{\tilde{u}} = \bar{u}e^{\gamma_5\vartheta/2}.$$

Taking into account that the Dirac bispinor amplitudes as usually Ref.[27] are normalized by invariant condition $\bar{u}u = 2m$. Hence we have

$$\bar{\tilde{u}}\tilde{u} = \bar{u}u = 2m. \quad (28)$$

By using (24),(25) and(28) we can also obtain

$$\bar{\tilde{u}}e^{-\gamma_5\vartheta}\gamma_\mu\tilde{u} = 2p_\mu.$$

Taking into account (14) and (21),(22) one can easily find

$$j_\mu = \frac{1}{2E}\tilde{u}\gamma_\mu\eta\tilde{u} = \{1, \mathbf{p}/E\},$$

whence it follows that the operator $\eta = e^{\gamma_5\vartheta}$ in full compliance with Mostafazadeh's result (see, for example Ref. [11],[10]) induces the inner product

$$\tilde{\psi}^*\eta\tilde{\psi} = 1,$$

for $\tilde{\psi} \neq 0$.

3 Dirac modified models with γ_5 -massive contributions in the external homogenies magnetic field

As it is known the wave Dirac equations provide a basis for relativistic quantum mechanics and quantum electrodynamics of spinor particles in external electromagnetic fields. Exact solutions of relativistic wave equation are referred to as one-particle wave functions which allow the development of the approach known as the Furry picture. This method incorporates study the interactions with the external field exactly, regardless of the field intensity(see Ref.[27]). Without the knowledge of exact solutions there is no regular methods of describing such interactions with an arbitrariness field explicitly. The physically most important exact solutions of the ordinary Dirac equations are: an electron in a Coulomb field, in a uniform magnetic field and in the field of a plane wave. In this connection, is of interest the investigating of non-Hermitian Dirac models which describe an alternative formulation of relativistic quantum mechanics where the Furry picture also may be realized.

Consider a uniform magnetic field $\mathbf{H} = (0, 0, H)$ directed along the x_3 axis ($H > 0$). The electromagnetic potentials are chosen in the gage Ref.[27]

$$A_0 = 0, \quad A_1 = 0, \quad A_2 = Hx_1, \quad A_3 = 0. \quad (29)$$

We can write the modified Dirac equations in the form

$$(\gamma_\mu \mathcal{P}^\mu - me^{\vartheta\gamma_5}) \tilde{\Psi} = 0, \quad (30)$$

were $\mathcal{P}^\mu = i\partial_\mu - eA_\mu$; $e = -|e|$ and γ -matrixes still chosen in the standard representation. In the field under consideration, the operators \mathcal{P}_0 , \mathcal{P}_2 and \mathcal{P}_3 are mutually commuting integrals of motion $[\mathcal{D}, \mathcal{P}_0] = 0, [\mathcal{D}, \mathcal{P}_2] = 0, [\mathcal{D}, \mathcal{P}_3] = 0$, where $\mathcal{D} = (\gamma_\mu \mathcal{P}^\mu - me^{\vartheta\gamma_5})$.

Let present the function $\tilde{\Psi}$ in the form

$$\tilde{\Psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} e^{-iEt}$$

and use Hamilton's form of Dirac equations

$$H\tilde{\psi} = E\tilde{\psi} \quad (31)$$

where

$$H = (\vec{\alpha}\mathbf{P}) + \beta m_1 + \beta\gamma_5 m_2.$$

It is useful to introduce the change of variables Ref.[27]

$$\psi_i(x_1, x_2, x_3) = e^{ip_2x_2 + ip_3x_3} \Phi_i(x_1),$$

where $i = 1, 2, 3, 4$. we can obtain the following system of equations:

$$(E \mp m_1)\Phi_{1,3} + iR_2\Phi_{4,2} - (p_3 \mp m_2)\Phi_{3,1} = 0; \quad (32)$$

where $R_2 = \left[\frac{\partial}{\partial x_1} + (p_2 + eH) \right]$;

$$(E \mp m_1)\Phi_{2,4} + iR_1\Phi_{3,1} + (p_3 \pm m_2)\Phi_{4,2} = 0. \quad (33)$$

Here $R_1 = \left[\frac{\partial}{\partial x_1} - (p_2 + eH) \right]$ and top mark relates to the components of the wave function with the first index, and the lower - to the components with the second index

Next convenient to go to the dimensionless variable

$$\rho = \sqrt{\gamma}x_1 + p_2/\sqrt{\gamma}, \quad (34)$$

where $\gamma = |e|H$, and equations (32),(33) take the form

$$(E \mp m_1)\Phi_{1,3} + i\sqrt{\gamma} \left(\frac{d}{d\rho} + \rho \right) \Phi_{4,2} - (p_3 \mp m_2)\Phi_{3,1} = 0; \quad (35)$$

$$(E \mp m_1)\Phi_{2,4} + i\sqrt{\gamma} \left(\frac{d}{d\rho} - \rho \right) \Phi_{3,1} + (p_3 \pm m_2)\Phi_{4,2} = 0. \quad (36)$$

General solution of this system can be represented in the form of the Hermite functions

$$u_n(\rho) = \left(\frac{\gamma^{1/2}}{2^n n! \pi^{1/2}} \right) e^{-\rho^2/2} H_n(\rho),$$

where $H_n(x)$ is standardizing the Hermite polynomials:

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2},$$

and $n = 0, 1, 2, \dots$. It should be noted that Hermite functions are satisfied to the recurrent relations:

$$\left(\frac{d}{d\rho} + \rho \right) u_n = (2n)^{1/2} u_{n-1}; \quad (37)$$

$$\left(\frac{d}{d\rho} - \rho \right) u_{n-1} = -(2n)^{1/2} u_n. \quad (38)$$

It is easy to see from (37),(38) that

$$\left(\frac{d}{d\rho} - \rho \right) \left(\frac{d}{d\rho} + \rho \right) u_n = -2n u_n$$

and hence (see, for example Ref.[27])

$$R_1 R_2 = -2\gamma n. \quad (39)$$

Substituting next in (35),(36)

$$\Phi = \begin{pmatrix} C_1 u_{n-1}(\rho) \\ iC_2 u_n(\rho) \\ C_3 u_{n-1}(\rho) \\ iC_4 u_n(\rho) \end{pmatrix},$$

one can find that coefficients C_i ($i = 1, 2, 3, 4$) are determined by algebraic equations

$$\begin{aligned}(E \mp m_1)C_{1,3} - (2\gamma n)^{1/2}C_{4,2} - (p_3 \mp m_2)C_{3,1} &= 0; \\ (E \mp m_1)C_{2,4} - (2\gamma n)^{1/2}C_{3,1} + (p_3 \pm m_2)C_{4,2} &= 0.\end{aligned}$$

The equality to zero of the determinant of this system leads to a spectrum of energy of the non-Hermitian Hamiltonian in the form

$$E = \pm \sqrt{m_1^2 - m_2^2 + 2\gamma n + p_3^2}, \quad (40)$$

where $n = 0, 1, 2, \dots$, and with take into account $m^2 = m_1^2 - m_2^2$, we can see the result, which also (see(13)) coincides with the eigenvalues of Hermitian Hamiltonian Ref.[27].

The coefficients C_i may be determined if one uses operator of polarization in the form of the third component of the polarization tensor in the direction of the magnetic field

$$\mu_3 = m_1\sigma_3 + \rho_2[\vec{\sigma}\vec{\mathcal{P}}]_3 \quad (41)$$

where matrices

$$\sigma_3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}; \quad \rho_2 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}.$$

It is easy to see, that bispinor C can be written as

$$\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} \cosh(\vartheta/2)\Phi_1 + \sinh(\vartheta/2)\Phi_3 \\ \cosh(\vartheta/2)\Phi_2 + \sinh(\vartheta/2)\Phi_4 \\ \sinh(\vartheta/2)\Phi_1 + \cosh(\vartheta/2)\Phi_3 \\ \sinh(\vartheta/2)\Phi_2 + \cosh(\vartheta/2)\Phi_4 \end{pmatrix}, \quad (42)$$

where

$$\begin{aligned}\Phi_1 &= \sqrt{1 + \zeta m/p_\perp} \sin(\pi/4 + \lambda/2) \\ \Phi_2 &= \zeta \sqrt{1 - \zeta m/p_\perp} \sin(\pi/4 - \lambda/2) \\ \Phi_3 &= \zeta \sqrt{1 + \zeta m/p_\perp} \sin(\pi/4 - \lambda/2) \\ \Phi_4 &= \sqrt{1 - \zeta m/p_\perp} \sin(\pi/4 + \lambda/2).\end{aligned}$$

Here $\mu_3\psi = \zeta k\psi$, $k = \sqrt{p_\perp^2 + m^2}$ and $\zeta = \pm 1$ that is corresponding to the orientation of the fermion spin: along (+1) or opposite (-1) to the magnetic field, and parameter λ obey to the condition $\cos \lambda = p_3/E$. The functions $\sinh(\vartheta/2)$ and $\cosh(\vartheta/2)$ are defined by relations (12).

4 Non-Hermitian modified Dirac-Pauli model in the magnetic field

In this section, we will touch upon a question of describing the motion of Dirac particles, if their own magnetic moment is different from the Bohr magneton. As it was shown by Schwinger Ref.[28], that the Dirac equation of particles in the external electromagnetic field A^{ext} taking into account the radiative corrections may be represented in the form

$$(\mathcal{P}\gamma - m)\Psi(x) - \int \mathcal{M}(x, y|A^{ext})\Psi(y)dy = 0, \quad (43)$$

where $\mathcal{M}(x, y|A^{ext})$ is the mass operator of fermion in external field. From equation (43) by means of expansion of the mass operator in series according to eA^{ext} with precision not over then linear field terms one can obtain the modified equation(see, for example, Ref.[27]). This equation preserves the relativistic covariance and consistent with the phenomenological equation of Pauli obtained in his early papers.

Now let us consider the model of massive fermions with γ_5 -extension of mass $m \rightarrow m_1 + \gamma_5 m_2$ taking into account the interaction of their charges and AMM with the electromagnetic field $F_{\mu\nu}$:

$$\left(\gamma^\mu \mathcal{P}_\mu - m_1 - \gamma_5 m_2 - \frac{\Delta\mu}{2} \sigma^{\mu\nu} F_{\mu\nu} \right) \tilde{\Psi}(x) = 0, \quad (44)$$

where $\Delta\mu = (\mu - \mu_0) = \mu_0(g - 2)/2$. Here μ - magnetic moment of a fermion, g - fermion gyromagnetic factor, $\mu_0 = |e|/2m$ - the Bohr magneton, $\sigma^{\mu\nu} = i/2(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$. Thus phenomenological constant $\Delta\mu$, which was introduced by Pauli, is part of the equation and gets the interpretation with the point of view quantum field theory.

The Hamiltonian form of (44) in the homogenies magnetic field is the following

$$i\frac{\partial}{\partial t}\tilde{\Psi}(r, t) = H_{\Delta\mu}\tilde{\Psi}(r, t), \quad (45)$$

where

$$H_{\Delta\mu} = \vec{\alpha}\vec{\mathcal{P}} + \beta(m_1 + \gamma_5 m_2) + \Delta\mu\beta(\vec{\sigma}\mathbf{H}). \quad (46)$$

Given the quantum electrodynamic contribution in AMM of an electron with accuracy up to e^2 order we have $\Delta\mu = \frac{\alpha}{2\pi}\mu_0$, where $\alpha = e^2 = 1/137$ - the

fine-structure constant and we still believe that the potential of an external field satisfies to the free Maxwell equations.

It should be noted that now the operator projection of the fermion spin at the direction of its movement - $\vec{\sigma} \cdot \vec{P}$ is not commute with the Hamiltonian (46) and hence it is not the integral of motion. The operator, which is commuting with this Hamiltonian remains μ_3 (see (41)). Subjecting the wave function $\tilde{\psi}$ to requirement to be eigenfunction of the operator polarization (41) and Hamilton operator (46) we can obtain:

$$\mu_3 \psi = \zeta k \psi, \quad \mu_3 = \begin{pmatrix} m_1 & 0 & 0 & \mathcal{P}_1 - i\mathcal{P}_2 \\ 0 & -m_1 & -\mathcal{P}_1 - i\mathcal{P}_2 & 0 \\ 0 & -\mathcal{P}_1 + i\mathcal{P}_2 & m_1 & 0 \\ \mathcal{P}_1 + i\mathcal{P}_2 & 0 & 0 & -m_1 \end{pmatrix}, \quad (47)$$

where $\zeta = \pm 1$ are characterized the projection of fermion spin at the direction of the magnetic field.

$$H_{\Delta\mu} \tilde{\psi} = E \tilde{\psi},$$

$$H_{\Delta\mu} = \begin{pmatrix} m_1 + H\Delta\mu & 0 & \mathcal{P}_3 - m_2 & \mathcal{P}_1 - i\mathcal{P}_2 \\ 0 & m_1 - H\Delta\mu & \mathcal{P}_1 + i\mathcal{P}_2 & -m_2 - \mathcal{P}_3 \\ m_2 + \mathcal{P}_3 & \mathcal{P}_1 - i\mathcal{P}_2 & -m_1 - H\Delta\mu & 0 \\ \mathcal{P}_1 + i\mathcal{P}_2 & m_2 - \mathcal{P}_3 & 0 & H\Delta\mu - m_1 \end{pmatrix}. \quad (48)$$

A feature of the model with γ_5 -mass contribution is that it may contain another any restrictions of mass parameters in addition to (7). Indeed while that for the physical mass m one may be constructed by infinite number combinations of m_1 and m_2 , satisfying to (2), however besides it need take into account the rules of conformity of this parameters in the Hermitian limit. Without this the developing of Non-Hermitian models may not be adequate. With this purpose one can determine an additional variable, which will depend on m_1 , m_2 and which would put an upper bound on the mass spectrum of particles. In particular, the simple a linear scheme may be constructed easy if one takes the obvious restriction for the mass spectrum of the fermions by using

$$m \leq M_1, \quad (49)$$

where M_1 is the auxiliary parameter which is equal to the value of mass parameter m_1 ($M_1 = m_1$). Using this approach we can describe in principle the whole spectrum of fermions when $m \leq M_1$ by means of defining appropriate values m_2 .

According to (2) one can obtain the expression

$$m = M_1 \sqrt{1 - m_2^2 / M_1^2}. \quad (50)$$

With the help (50) we can see, that when the mass m_2 is increased, the values of the physical mass tends to zero (see Fig.1). The equality of parameters $m_2 = m_1$ corresponds to the case of massless fermions. But it should be noted that in this linear model the Hermitian limit $m_2 \rightarrow 0$ may be reached only in the case of the particles with maximal mass $m = M_1$. At the same time, *the Hermitian limit is absent for all other mass values.*

Thus the procedure of limitations of the physical mass spectrum by the inequality $m \leq M_1$ has the essential drawback since in this frame is not possible to describe all ordinary fermions, respecting to the Principle of Conformity, except, $m = M_1$.

These considerations make search in the frame of mass restriction of $m \leq m_1$ the existence of more complicated non-linear dependence of limiting mass value

$$m \leq M(m_1, m_2), \quad (51)$$

which meets the requirements of the Principle of Conformity:

i) The Dirac limit must exist for all ordinary fermions for which the condition (51) is satisfied.

ii) In Hermitian limit the parameter m_1 must coincide with any physical mass m .

The fulfillment of these conditions lets to find the most appropriate scheme restriction of the mass fermions, for which exist the consistency with the ordinary Dirac theory. Possibly explicit expression for $M(m_1, m_2)$, may be obtained from the **simple mathematical theorem**: *the arithmetical average of two non-negative real numbers a and b which always is greater than or equal to the geometrical mean of the same numbers.* Really, let $a = m^2$ and $b = m_2^2$ then using

$$\frac{m^2 + m_2^2}{2} \geq \sqrt{m^2 \cdot m_2^2}$$

and substitution (2), we can get the inequality

$$m \leq m_1^2 / 2m_2 = M(m_1, m_2). \quad (52)$$

Values of M is now defined by two parameters m_1, m_2 . And in the Hermitian limit, when $m_2 \rightarrow 0$ the value of the maximal mass M tends to infinity. It

is very important that in this limit the restriction of mass values completely disappear. In such a way one can demonstrate a natural transition from the Modified Model to the Standard Model (SM), which contains any values of the physical mass m .

Using (2) and expression (52) we can also obtain the system of two equations

$$\begin{cases} m = m_1^2 - m_2^2 \\ M = m_1^2/2m_2 \end{cases} \quad (53)$$

The solution of this system relative to the parameters m_1 and m_2 may be represented in the form

$$m_1^\mp = \sqrt{2}M\sqrt{1 \mp \sqrt{1 - m^2/M^2}}; \quad (54)$$

$$m_2^\mp = M \left(1 \mp \sqrt{1 - m^2/M^2} \right). \quad (55)$$

It is easy to verify that the obtained values of the mass parameters satisfy the conditions (2) and (7) regardless of which the sign will be chosen. Besides formulas (54),(55) in the case of the upper sign are agreed with conditions $m_2 \rightarrow 0$ and $m_1 \rightarrow m$ when $M \rightarrow \infty$, i.e. when a the Hermitian limit is exist. However, if one choose a lower sign (i.e. for the m_1^+ and m_2^+) such limit is absent. Thus we can see that the nonlinear scheme of mass restrictions (see (52)) additionally contains the solutions satisfying to the some new particles. However this solutions should be considered only as an indication of the principal possibility of the existence of such particles. In this case, as follows from (54),(55) for each ordinary particles may be exist some new partners, possessing the same mass and a number of another unique properties.¹

¹As the exotic particles do not agree in the "flat limit" with the ordinary Dirac expressions then one can assume that in this case we deal with a description of some new particles, properties of which have not yet been studied. This fact for the first time has been fixed by V.G.Kadyshevsky in his early works in the geometric approach to the development of the quantum field theory with a fundamental mass" Ref.[29] in curved de-Sitter momentum space. Besides in Ref.[30],[31] it was noted that the most intriguing prediction of the new approach is the possible existence of exotic fermions with no analogues in the SM, which may be candidates for constituents of dark matter .

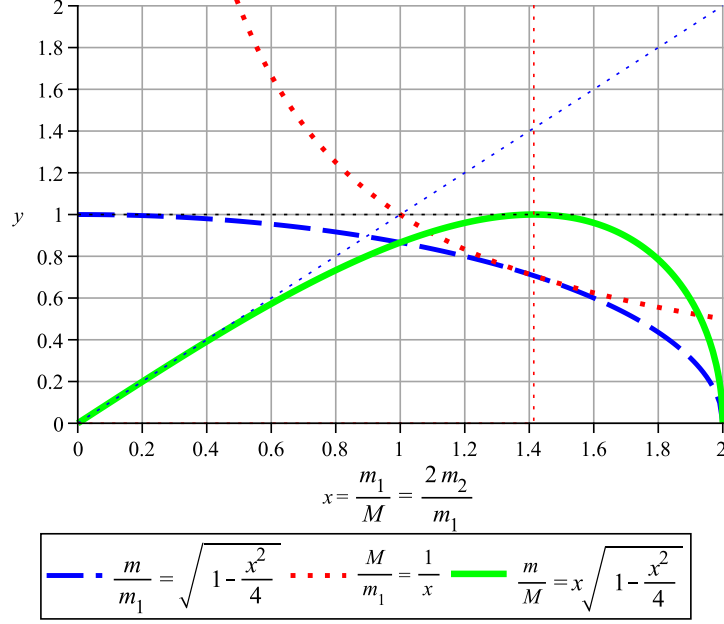


Figure 1: Dependence of m/M , M/m_1 , and m/m_1 on the parameter $x = m_1/M = 2m_2/m_1$

Let's consider the "normalized" parameter of the modified model with the maximal mass M :

$$x = \frac{m_1}{M} = \frac{2m_2}{m_1} \quad (56)$$

and using (50) we can obtain

$$\frac{m_2}{M} = \frac{x^2}{2} \quad (57)$$

$$\frac{m}{M} = x\sqrt{1 - x^2/4} \quad (58)$$

At Fig.1 one can see the dependence of the normalized parameters m/m_1 , M/m_1 and m/M on the relative parameter $x = m_1/M = 2m_2/m_1$. In particular, the maximum value of the particle mass $m = M$ is achieved at the ratio of the subsidiary masses is equal to $m_2 = m_1/\sqrt{2}$. Till to this value for each mass of ordinary particles, one can find the parameters m_1 and m_2 , for which a limit transition to regular theory Dirac exist. Further increasing of m_2 , leads to the descending branch of the m/M , where the Dirac limit is

not exist and at the point $m_2 = m_1$ the value of m is equal to zero. Thus, it is the region $m_1 > \sqrt{2}m_2$ ($m_2 > M$) corresponds to the description of the "exotic particles", for which there is not transition to Hermitian limit.

Thus, in the frame of the known inequality (7) we can see three specific sectors of unbroken \mathcal{PT} -symmetry of the non-Hermitian Hamiltonian (5) in the plane $\nu_1 = m_1/M, \nu_2 = m_2/M$. Thus the plane ν_1, ν_2 may be divided by the three groups of the inequalities:

$$\begin{aligned} I. \quad & \nu_1/\sqrt{2} \leq \nu_2 \leq \nu_1, \\ II. \quad & -\nu_1/\sqrt{2} < \nu_2 < \nu_1/\sqrt{2}, \\ III. \quad & -\nu_1 \leq \nu_2 \leq -\nu_1/\sqrt{2}, \end{aligned}$$

It is very important that only the region *II.* corresponds to the description of ordinary particles, while *I.* and *III.* define the description of some as yet unknown particles. This conclusion is not trivial, because in contrast to the geometric approach, where the emergence of new unusual properties of particles associated with the presence in the theory a new degree of freedom (sign of the fifth component of the momentum $\varepsilon = p_5/|p_5|$ [30]), in the case of a simple extension of the free Dirac equation due to the additional γ_5 -mass term, the satisfactory explanation of this fact is not there yet.

Then we can establish the limits of change of parameters. As it follows from the (54),(55) the limits of variation of parameters m_1 and m_2 are the following:

$$m \leq m_1 \leq 2M; -2M \leq m_2 \leq 2M. \quad (59)$$

In the areas of change of these parameters a point exists in which we have

$$m_1 = \sqrt{2}M; m_2 = M. \quad (60)$$

In this point the physical mass m reaches its maximum value $m = M$ (see Fig.1) that corresponds to the mass of the "maximon" Ref.[29].

Performing calculations here in many ways reminiscent of similar calculations carried out in the previous section. In a result of modified Dirac-Pauli equation one can also find *the exact solution for energy spectrum*:

$$E(\zeta, p_3, 2\gamma n, H) = \pm \sqrt{p_3^2 - m_2^2 + \left[\sqrt{m_1^2 + 2\gamma n} + \zeta \Delta \mu H \right]^2} \quad (61)$$

and for eigenvalues of the operator polarization μ_3 we can write in the form

$$k = \sqrt{m_1^2 + 2\gamma n}. \quad (62)$$

It is easy to see that in the case $\Delta\mu = 0$ from (61) one can obtain the expression (40). Besides it should be emphasized that the expression analogical to (61), in the frame of ordinary Dirac-Pauli approach one can obtain putting $m_2 = 0$ and $m_1 = m$:

$$E(\zeta, p_3, 2\gamma n, H) = \pm \sqrt{p_3^2 + \left[\sqrt{m^2 + 2\gamma n} + \zeta \Delta\mu H \right]^2}. \quad (63)$$

Note that in the paper Ref.[32] was previously obtained result analogical to (63) by means of using the ordinary Dirac-Pauli approach. Direct comparison of modified formula (61) in the Hermitian limit with the result Ref.[32] shows their coincidence. It is easy to see that the expression (61) contains dependence on parameters m_1 and m_2 separately, which are not combined into *a mass of particles* $m = \sqrt{m_1^2 - m_2^2}$ that essentially differs from the examples which were considered in previous sections **2, 3**.

Thus, in contrast to (13) and (40) here the calculation of interaction AMM of fermions with the magnetic field allow to put the question about the possibility of experimental studies of the effects of γ_5 -extensions of a fermion mass. In particular if to suggest that $m_2 = 0$ and hence $m_1 = m$, we obtain as noted earlier, the Hermitian limit. But taking into account the expressions (54) and (55) we obtain that the energetic spectrum (61) is expressed through the fermion mass m and the value of the maximal mass M . Thus, taking into account that the interaction AMM with magnetic field removes the degeneracy on spin variable, we can obtain the energy of the ground state ($\zeta = -1$) in the form

$$E(-1, 0, 0, H, x) = m \sqrt{-\left(\frac{1 \mp \sqrt{1-x^2}}{x}\right)^2 + \left(\frac{\sqrt{2}\sqrt{1 \mp \sqrt{1-x^2}}}{x} - \frac{\Delta\mu H}{m}\right)^2}, \quad (64)$$

where $x = m/M$ and the upper sign corresponds to the ordinary particle and the lower sign defines their "exotic" partners.

Through decomposition of functions ${}^-m_1$ and ${}^-m_2$ we can obtain

$${}^-m_1/m = \begin{cases} 1 + \frac{x^2}{8} + \frac{7x^4}{128}, & x \ll 1 \\ \frac{\sqrt{2}}{x} & x \rightarrow 1 \end{cases} \quad {}^-m_2/m = \begin{cases} \frac{x}{2} + \frac{x^3}{8} + \frac{x^5}{16}, & x \ll 1 \\ \frac{1}{x} & x \rightarrow 1 \end{cases} \quad (65)$$

Similarly, for ${}^+m_1$ and ${}^+m_2$ we have

$${}^+m_1/m = \begin{cases} \frac{2}{x} - \frac{x}{4} - \frac{5x^3}{64}, & x \ll 1 \\ \frac{\sqrt{2}}{x} & x \rightarrow 1 \end{cases} \quad {}^+m_2/m = \begin{cases} \frac{2}{x} - \frac{x}{2} - \frac{x^3}{8}, & x \ll 1 \\ \frac{1}{x} & x \rightarrow 1 \end{cases} \quad (66)$$

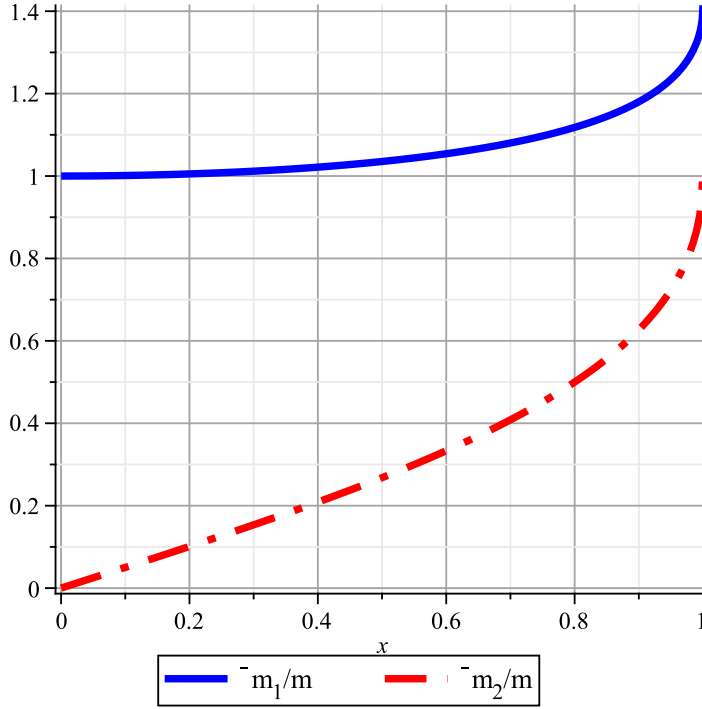


Figure 2: The dependence of parameters ${}^-m_1/m, {}^-m_2/m$ on the $x = m/M$.

Let us now turn to a more detailed consideration of the fermion energy in the ground state in the external field. As follows from (65) and (66) function (61) not trivial depends on the parameters $x = m/M$ and H . For reasons outlined above, the effect of magnetic field on the energy state of the ordinary fermion with the small mass $x \ll 1$ (see Fig.2) we can obtain in the form

$$E(-1, 0, 0, H, x) = m \sqrt{1 - \frac{\Delta\mu}{\mu_0} \frac{H}{H_c} (1 + x^2/8 + 7x^4/128) + \left(\frac{\Delta\mu H}{m} \right)^2}, \quad (67)$$

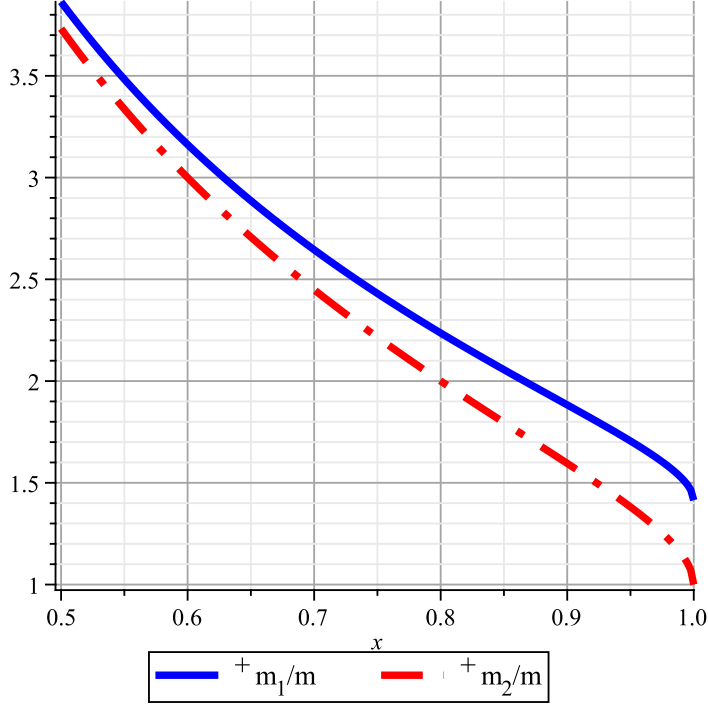


Figure 3: The dependence of parameters $+m_1/m, +m_2/m$ on the $x = m/M$.

where $H_c = m^2/e$.

On the other hand, for the case of the "exotic" particles in the similar limit $x \ll 1$ the result is significantly different (see (66) and Fig.3)

$$E(-1, 0, 0, H, x) = m \sqrt{1 - \frac{\Delta\mu}{\mu_0} \frac{2H}{xH_c} + \left(\frac{\Delta\mu H}{m}\right)^2}. \quad (68)$$

From (68) one can see that the field corrections in this case are substantially increased as $1/x = M/m \gg 1$.

If the use the limit of the large mass $x \rightarrow 1$ we get the matching results as for traditional and exotic particles. Thus, by combining these results, we can write

$$E(-1, 0, 0, H, x) = m \sqrt{1 - \frac{\Delta\mu}{\mu_0} \frac{\sqrt{2}H}{xH_c} + \left(\frac{\Delta\mu H}{m}\right)^2}. \quad (69)$$

One can also see from (65),(66) that the changes of the parameters ${}^\mp m_1$ and ${}^\mp m_2$ occur by such a way that in the point $x = 1$ ($m = M$) the branches of ordinary and exotic particles are crossed. At Fig.2 and Fig.3 dependencies of m_1^\mp/m and m_2^\mp/m on the parameter $x = m/M$ are represented and one may clearly see the justice of this fact.

As the equation (44) and following from it formulas (61),(67),(68) would fair for any intensities magnetic field, it is easy to see with the values in

$$H \sim \frac{\mu_0}{\Delta\mu} \frac{m}{m_1} H_c, \quad (70)$$

we would obtain $E_0 \sim 0$. Hence, in the intensive magnetic fields accounting of the vacuum magnetic moment can lead to a substantial change of borders of the energetic spectrum between the fermion and anti fermion states. Notice once more that a considerable increasing of this amendment is connected with the possible contributions from so-called exotic particles.

5 Summary and Conclusions.

In the researches, presented in the previous sections, shown that the Dirac Hamiltonian of a particle with γ_5 - dependent mass term is non-Hermitian, and has the unbroken \mathcal{PT} - symmetry in the area $m_1^2 \geq m_2^2$, which has three the subregion. Indeed with the help of the algebraic transformations we obtain a number of the consequence of the relation (2). In particular there is the restriction of the particle mass in this model: $m \leq M$, were $M = m_1^2/2m_2$. Outside of this area the \mathcal{PT} - symmetry of the modified Dirac Hamiltonians is broken and the inequality $m \leq M$ may be considered as a *new wording of the conditions of unbroken \mathcal{PT} - symmetry*.

In addition, pay attention, that the introduction of limitation of the mass spectrum, on the basis of a geometric approach to the development of the modified QFT (see, for example [30],[31]), also leads to appearance of non-Hermitian \mathcal{PT} -symmetric Hamiltonians in the fermion sector of the model with the Maximal Mass. On the other hand, it was shown by us, that non-Hermitian \mathcal{PT} -symmetric algebraic approach with γ_5 - mass term, may be considered as a condition of occurrence of the analogical auxiliary mass parameter of the model.

In particular, this applies to the modified Dirac equation in which was produced the substitution $m \rightarrow m_1 + \gamma_5 m_2$. Into force of the ambiguity of

the definition of the parameter m depending on m_1, m_2 , the basic inequality $m_1 \geq m_2 \geq 0$ contains description of particles of two species. In the first case, it is ordinary particles, when mass parameters are limited by the terms

$$0 \leq m_2 \leq m_1/\sqrt{2}. \quad (71)$$

In the second area we are dealing with so-called exotic partners of ordinary particles, for which is still accomplished (7), but one can write

$$m_1/\sqrt{2} \leq m_2 \leq m_1. \quad (72)$$

Intriguing difference of the second type particles from traditional fermions is that they are described by the different modified Dirac equations. So, if in the first case(71), the equation of motion under the transition $M \rightarrow \infty$ leads to the standard Dirac equation, but in the second case such a transition is not there.

Thus, it is shown that the main progress, is obtained by us in the algebraic way of the construction of the fermion model with γ_5 -mass term is consists of describing of the new energetic scale, which is defined by the parameter $M = m_1^2/2m_2$. This value on the scale of the masses is a point of transition from the ordinary particles to exotic. Furthermore, description of the exotic particles in the algebraic approach are turned out essentially the same as in the model with a maximal mass, which was investigated by V.G.Kadyshevsky with colleagues on the basis of geometrical approach (see, for example Ref.[29]-[31]).

We have presented a number of examples of non-Hermitian models with γ_5 -extension mass in relativistic quantum mechanics including in presence of external electromagnetic field for which the Hamiltonian H has a real spectrum. Although the energy spectra of the fermions in some cases were makes them indistinguishable from the spectrum of corresponding Hermitian Hamiltonian H_0 we found example, in which the energy of fermions is clearly dependent on non-Hermitian characteristics. We are talking about the consideration of the interaction of AMM of fermions with a magnetic field. In this case we obtained the exact solution for the energy of fermions (see (61)).

It should be noted that the formula (61) is a valid not only for charged fermions, but and for the neutral particles possessing AMM. In this case one must simply replace the value of quantized transverse momentum of a charged particle in a magnetic field on the ordinary value $2\gamma n \rightarrow p_1^2 + p_2^2$.

Thus, for the case of ultra cold polarized ordinary electronic neutrino with precision not over then linear field terms we can write

$$E(-1, 0, 0, H, m_{\nu_e}/M \rightarrow 0) = m_{\nu_e} \sqrt{1 - \frac{\mu_{\nu_e}}{\mu_0} \frac{H}{H_c}}. \quad (73)$$

However, in the case of exotic electronic neutrino we have

$$E(-1, 0, 0, H, m_{\nu_e}/M) = m_{\nu_e} \sqrt{1 - \frac{\mu_{\nu_e}}{\mu_0} \frac{2MH}{m_{\nu_e} H_c}}. \quad (74)$$

It is well known [34],[35] that in the minimally extended SM the one-loop radiative correction generates neutrino magnetic moment which is proportional to the neutrino mass

$$\mu_{\nu_e} = \frac{3}{8\sqrt{2}\pi^2} |e| G_F m_{\nu_e} = (3 \cdot 10^{-19}) \mu_0 \left(\frac{m_{\nu_e}}{1eV} \right), \quad (75)$$

where G_F -Fermi coupling constant and μ_0 is Bohr magneton. However, so far, the most stringent laboratory constraints on the electron neutrino magnetic moment come from elastic neutrino-electron scattering experiments: $\mu_{\nu_e} < (1.5 \cdot 10^{-10}) \mu_0$ [36]. Besides the discussion of problem of measuring the mass of neutrinos (either active or sterile) show that for an active neutrino model we have $\sum m_\nu = 0.320eV$, whereas for a sterile neutrino $\sum m_\nu = 0.06eV$ [37].

One can also estimate the change of the border of region of unbroken \mathcal{PT} -symmetry due to the shift of the lowest-energy state in the magnetic field. Using formulas (73) and (74) we obtain correspondingly regions of undisturbed \mathcal{PT} -symmetry in the form

$$H_{\nu_e\text{-ordinary}} \leq \frac{\mu_0}{\mu_{\nu_e}} H_c; \quad (76)$$

$$H_{\nu_e\text{-exotic}} \leq \frac{m_{\nu_e} \mu_0}{2M \mu_{\nu_e}} H_c. \quad (77)$$

Indeed let us take the following parameters of neutrino: the mass of the electronic neutrino is equal to $m_{\nu_e} = 1eV$ and magnetic moment equal to (75). If we assume that the values of mass and magnetic moment of exotic

neutrino identical to parameters of ordinary neutrinos, we can obtain the estimates of the border area undisturbed \mathcal{PT} symmetry for (76) in the form

$$H_{\nu_e-ordinary}^{cr} = \frac{\mu_0}{\mu_{\nu_e}} H_c \sim 10^{32} Gauss. \quad (78)$$

However in the case (77) the situation may change radically

$$H_{\nu_e-exotic}^{cr} = \frac{\mu_0}{\mu_{\nu_e}} \frac{m_{\nu_e}}{2M} H_c \sim 10^4 Gauss. \quad (79)$$

In comparison with (78) where the experimentally possible field corrections are extremely small one can see that the critical value of magnetic field (79) is attainable in the sense of ordinary terrestrial experiments. In (78) and (79) we used the values of quantum-electrodynamic constant $H_c = 4.41 \cdot 10^{13} Gauss$ and the Planck mass $M = m_{Planck} \simeq 10^{19} GeV$. We do not know if there is an upper limit to spectrum masses of elementary particles Ref.[38] and many are skeptical about the significance of the Planck mass, but experimental studies of this thesis at high energies hardly today may even be discussed. However contemporary precision of alternative laboratory measurements at low energy in the magnetic field may in principle allow to achieve the required values for exotic particles in the near future. Thus, the obtained formulas (74)-(79) ensures possibility not only to be convinced in the existence of the Maximal Mass but and in reality of the so-called *exotic particles*, because this phenomena are inextricably related.

Note also that intensive magnetic fields exist near and within a number of space objects. So, the magnetic fields intensity of the order of $10^{12} \div 10^{13} Gauss$ observed near pulsars. Here also may be included the recent opening of such objects as sources soft repeated gamma-ray burst and anomalous x-ray pulsars. For them magneto-rotational models are proposed, and they were named as magnetars. It was showed that for such objects achievable magnetic fields with intensity up to $10^{15} Gauss$. It is very important that the share of magnetars in the General population of neutron stars reaches 10%. In this regard, we note that the processes with the participation of ordinary neutrinos and especially of their possible "exotic partners in the presence of such strong magnetic fields can have a significant influence on the processes which may determine the evolution of astrophysical objects.

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